location determined by using the velocity defect (see Ref. 6). The data indicate that the growth of the wake is small in the laminar region, increases at transition, and that it finally levels off to the growth rate found previously for incompressible fully developed turbulent flows. This is particularly clear in the data of Walsh,[3] Behrens et al.,[5] and Demetriades.[6,7] In summary, Fig. 3 shows that the wake growth rate variation is a better indicator of the onset of fully developed turbulent flow than similarity variables used previously.

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# Exact Similar Solution for an Axisymmetric Laminar Boundary Layer on a Circular Cone


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ALTHOUGH approximate similar solutions are available for a laminar boundary layer on a circular cone,[1] the mainstream velocity distribution is then linked to the vertex angle of the cone. However, an exact solution of the Navier-Stokes equations with an explicit expression for $c_f$ is obtained when, for any vertex angle, the mainstream velocity is inversely proportional to the distance from the vertex.

Using spherical polar coordinates centered at the vertex of the cone, a velocity component $u$ is defined as positive in the direction of increasing radial distance $r$; $v$ is considered positive in the direction of increasing $\theta$ (the semi-angle of the cone being $\theta_0$); there is no circumferential velocity. An irrotational main stream and constant density $\rho$ and kinematic viscosity $\nu$ are assumed.

The Navier-Stokes equation[2] for the $\theta$ direction is multiplied by $r$, then differentiated with respect to $r$ and subtracted from the $\theta$ derivative of the equation for the $r$ direction. A stream function $\psi$ is introduced, defined by

$$u = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}; \quad v = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

If the resulting equation is to yield similar solutions and the transverse curvature of the surface is accounted for, the boundary-layer thickness $\delta$ at a particular position must be proportional to the radius of the surface there, that is to $r\sin\theta_0$ (unless $\theta_0 = \pi/2$, in which case the surface has no curvature). Hence $\delta/r = $ constant and so $\delta$ may be expressed simply in terms of $\theta - \theta_0$; that is, $\theta$ may be regarded as the similarity variable, and we set $\psi = \nu\ell f(\theta)$, where $\ell$ is a function of $r$ only.

The equation of motion then becomes

$$\begin{aligned}
&\frac{2c}{r^2 s^3}\ell_r \ell_{rr} f^2 - ff'\left\{\frac{1+2c^2}{r^4 s^4}\ell\ell_r + \frac{2}{r^3 s^2}\ell\ell_{rr}\right. \\
&\left. - \frac{1}{r^2 s^2}\ell\ell_{rrr} + \frac{1}{r^2 s^2}\ell_r \ell_{rr}\right\} + \frac{3c}{r^4 s^3}\ell\ell_r ff'' - \frac{1}{r^4 s^2}\ell\ell_r ff''' \\
&+ (f')^2\left\{\frac{4c}{r^5 s^3}\ell^2 - \frac{c}{r^4 s^3}\ell\ell_r\right\} + f'f''\left\{\frac{1}{r^4 s^2}\ell\ell_r - \frac{4}{r^5 s^2}\ell^2\right\} \\
&= -\frac{1}{s}\ell_{rrrr} f + f'\left\{\frac{c(6s^2+3)\ell}{r^4 s^4} - \frac{4c\ell_r}{r^3 s^2} + \frac{2c\ell_{rr}}{r^2 s^2}\right\} \\
&+ f''\left\{\left(\frac{5c^2-8}{r^4 s^3}\right)\ell + \frac{4\ell_r}{r^3 s} - \frac{2\ell_{rr}}{r^2 s}\right\} + \frac{2c\ell}{r^4 s^2}f''' - \frac{\ell}{r^4 s}f'''' \qquad (1)
\end{aligned}$$

where $c = \cos\theta$, $s = \sin\theta$, the primes denote differentiation with respect to $\theta$, and the suffixes denote differentiation with respect to $r$.

To make the coefficients of $f'''$ and $f''''$ independent of $r$, the equation must be multiplied by $r^4/\ell$. Then the coefficient of $ff'''$, for example, becomes $-\ell_r/s^2$. Consequently, similar solutions require $\ell_r$ to be independent of $r$. The coefficient of $(f')^2$ becomes

$$\frac{4c\ell}{s^3 r} - \frac{c}{s^3}\ell_r$$

and so $\ell/r$ must also be independent of $r$. Thus $\ell \propto r$, and, as the coefficient of proportionality may be absorbed into the definition of $f$, we set $\ell = r$. Equation (1) therefore becomes

$$\begin{aligned}
&3\frac{c}{s^2}(f')^2 - \frac{3}{s}f'f'' - \left(\frac{1+2c^2}{s^3}\right)ff' + \frac{3c}{s^2}ff'' - \frac{1}{s}ff''' \\
&= \frac{c}{s^3}(2s^2+3)f' + \left(\frac{c^2-4}{s^2}\right)f'' + \frac{2c}{s}f''' - f''''
\end{aligned}$$

Changing the variable from $f(\theta)$ to $F(c)$ yields

$$3F'F'' + FF''' = 4cF''' - (1-c^2)F'''' \qquad (2)$$

where the primes now denote differentiation with respect to $c$.

Equation (2) is essentially that obtained by Morgan[3] for flow between two coaxial cones with a common vertex, but he did not consider the boundary-layer problem.


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## Boundary Conditions

The component of vorticity about an axis perpendicular to the $r-\theta$ plane is[2]

$$\frac{1}{r}\frac{\partial}{\partial r}(rv)-\frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{c}{r^2s^2}\nu f'+\frac{1}{r^2s}\nu f''=\frac{\nu s}{r^2}F''$$

For this to be zero at all values of $\theta$ in the main stream, $F'=N$ and $F=Nc+M$, where $M$ and $N$ are constants. Therefore $F'\to N$ as the irrotational mainstream is approached.

In the mainstream, where $u=U_m$, say,

$$u=-\frac{1}{r^2s}\frac{\partial\psi}{\partial\theta}=-\frac{\nu}{rs}f'=\frac{\nu}{r}F'=\frac{\nu N}{r}$$

Consequently $N=U_m r/\nu$, the local Reynolds number, and the pressure gradient must be such that $U_m \propto 1/r$. At the solid surface $F'=0$ and $F=\zeta\sin\theta_0$ where $\zeta=rv_0/\nu$ and $v_0$ represents the velocity through the surface.

## Solution of the Equation

Equation (2) may be integrated to

$$FF'=-(1-c^2)F''-2F+N(N+2)c+(N+2)M \tag{3}$$

and then to

$$\tfrac{1}{2}F^2=-(1-c^2)F'-2cF+\tfrac{1}{2}N(N+2)c^2$$
$$+(N+2)Mc+\tfrac{1}{2}M^2+N \tag{4}$$

the outer boundary conditions being used to determine the integration constants. Substituting into Eq. (4) the values applicable at the solid surface and solving for $M$, we obtain

$$M=-(N+2)c_0\pm\sqrt{-2Ns_0^2+(2c_0+\zeta s_0)^2} \tag{5}$$

where $c_0=\cos\theta_0$ and $s_0=\sin\theta_0$.

The local skin-friction coefficient $c_f$ may be calculated from Eq. (3), which gives

$$F''(c_0)=s_0^{-2}\{-2\zeta s_0+N(N+2)c_0+(N+2)M\}$$

It may be shown that the rate of shear $G=(u_\theta-v)/r$ and so $c_f$ is

$$\frac{\nu|G|_{\theta=\theta_0}}{\tfrac{1}{2}U_m^2}=\frac{2\nu}{U_m^2}\left|\frac{\nu f''}{r^2s}+\frac{\nu f}{r^2s}\right|_{\theta=\theta_0}=-\frac{2}{N^2}\left|F''s+\frac{F}{s}\right|_{\theta=\theta_0}$$

i.e.,

$$c_f=\frac{2}{N^2}\{\zeta+(N+2)\cot\theta_0[2\pm\sqrt{(2+\zeta\tan\theta_0)^2-2N\tan^2\theta_0}]\} \tag{6}$$

The alternative signs before the square root in Eq. (6) each correspond to genuine solutions of the equation of motion, but monotonic velocity profiles are obtainable only with large positive values of $N$ (i.e., flow away from the vertex).[4] Moreover, suction is required whether the flow is inside or outside the cone.

Equation (6) is represented in Fig. 1. For flow inside the cone (i.e., $\theta<\theta_0$), $\zeta$ must always exceed

$$\left(\frac{N+2}{N+3}\right)\left[-2\cot\theta_0+\sqrt{4\cot^2\theta_0+2N\left(\frac{N+3}{N+1}\right)}\right]$$

and the negative sign in Eq. (6) is used. For flow outside the cone, $(\theta>\theta_0)$, $-\zeta$ must either

a) exceed

$$2\cot\theta_0+\sqrt{2N}\ \text{if}\ 2(N+1)\cot\theta_0-\sqrt{2N}\geq 0$$

or

b) exceed

$$\left(\frac{N+2}{N+3}\right)\left[2\cot\theta_0+\sqrt{4\cot^2\theta_0+2N\left(\frac{N+3}{N+1}\right)}\right]$$

$$\text{if}\ 2(N+1)\cot\theta_0-\sqrt{2N}\leq 0$$

In each case the positive alternative sign in Eq. (6) is taken. It may be shown readily from Eq. (6) that $|c_f|$ increases with the amount of suction and with increasing concave curvature, and decreases with increasing convex curvature.

The "displacement effect" of the boundary layer is incorporated in the value of $N$ since $U_m$ represents the actual mainstream velocity rather than the velocity found in the hypothetical absence of a boundary layer. The displacement thickness corresponds to $\theta^*-\theta_0$ and may be obtained from

$$\cos\theta^*=\frac{1}{N}[\zeta s_0+(N+2)c_0\mp\sqrt{(2c_0+\zeta s_0)^2-2Ns_0^2}]$$

the negative alternative sign being used when $\theta^*<\theta_0$, and the positive sign when $\theta^*>\theta_0$.

If $\theta_0=\pi/2$ the surface has no curvature, and the restriction that the boundary-layer thickness be proportional to the radius of curvature is thus removed. As shown by Crabtree et al.,[5] this flow—radial flow over a plane—is described by the Falkner-Skan equation. That equation, however, is only approximate (involving the assumption that the boundary-layer thickness is very small compared with $r$) and for $U_m r=$ constant, yields

$$c_f=\frac{2}{N}\sqrt{\zeta^2-2N}$$

The present exact analysis gives

$$c_f=\frac{2}{N^2}\{-|\zeta|+(N+2)\sqrt{\zeta^2-2N}\}$$

which reduces to the Falkner-Skan result when $N$ is large and $|\zeta|$ is appreciably more than the minimum.

Since the conditions described by the foregoing results would be difficult (though, in the presence of another boundary, not impossible) to achieve in practice, experimental verification is hardly to be hoped for, but the results may well be of value in checking numerical techniques.

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